PERGAMON

International Journal of Solids and Structures 37 (2000) 197–214

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Stochastic mechanics

Mircea Grigoriu*

School of Civil and Environmental Engineering, Cornell University, Ithaca, NY 14853, USA

Abstract

The properties of many physical systems and/or the actions on these systems exhibit notable random fluctuations that cannot be captured by deterministic models. Examples are used to demonstrate the need for probabilistic models to represent systems and inputs depending on uncertain parameters. The displacement, stress, strain, damage, and other output measures of these systems are stochastic so that their properties cannot be obtained by the deterministic methods of classical solid mechanics. The methods of stochastic mechanics are needed for solution. Following an introduction and a section on elementary concepts of probability theory, the four possible combinations of deterministic/stochastic system and input are examined in separate sections. The case of deterministic systems and inputs is discussed to demonstrate the potential of the stochastic methods for solving classical problems of solid mechanics. The case of deterministic operators and random inputs is focused on random vibration problems because these problems constitute a major topic of stochastic mechanics. The last two sections deal with problems defined by stochastic operators and deterministic or stochastic inputs. All the sections on stochastic mechanics review some of the most notable past accomplishments, outline current research trends on both analytical and numerical solutions, and illustrate some of these research trends by simple examples.

Keywords: Homogenization; Probability theory; Random vibration; Random walk; Reliability; Solid mechanics; Stochastic mechanics

1. Introduction

The properties of many physical systems and/or the input to these systems exhibit complex random fluctuations that cannot be characterized completely by deterministic models. Probabilistic models are needed to (1) quantify the uncertainty in these properties, (2) develop realistic representations of the output and the damage state of these systems, and (3) obtain rational and safe designs. Figs. 1 and 2 illustrate the uncertainties in the system and the input, respectively. Fig. 1 shows histograms of the Euler

* Tel.: +1-607-255-3334; fax: +1-607-255-4828.
E-mail address: mdg12@cornell.edu (M. Grigoriu)

0020-7683/00/$ - see front matter
PII: S0020-7683(99)00088-8

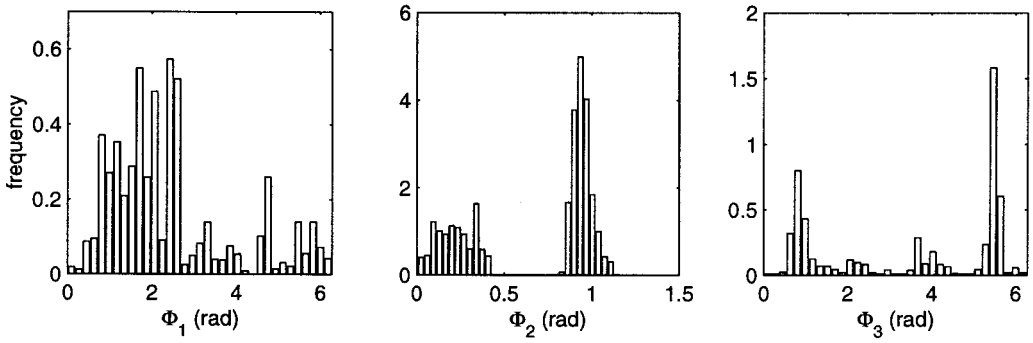

Fig. 1. Lattice orientation for aluminum grains.

angles $\mathbf{\Phi}=(\Phi_1, \Phi_2, \Phi_3)$ of the atomic lattice in aluminum crystals (Arwade et al., 1998). These histograms give the range of values of $\mathbf{\Phi}$ and the likelihood of these values. The atomic lattice orientation exhibits a complex spatial variation that cannot be represented adequately by deterministic models and may constitute one of the controlling factors of fatigue life for metals. Fig. 2 shows a time record of the coastal wave height measured in meters at Duck, North Carolina and the histogram of a scaled version of this record. The solid line is the Gaussian density with the mean and variance of the scaled record. The differences between this line and the histogram demonstrate that the wave height record is not Gaussian. Wind pressures, seismic ground acceleration, aerodynamic forces, and other actions have similar features so that they need to be characterized by probabilistic models (Grigoriu, 1995).

Classical deterministic methods can provide only crude approximations for the response and the evolution of the damage state of systems subjected to inputs with uncertain properties. The analysis of these systems requires specialized techniques that account for the uncertainty in both the system and the input and characterize the system output by probabilistic models. A main objective of stochastic mechanics is the study of the input–output relationship for systems and inputs with uncertain properties. Table 1 gives relevant examples of stochastic mechanics problems for all combinations of input and system properties. The algebraic or differential equations defining the evolution of a physical system

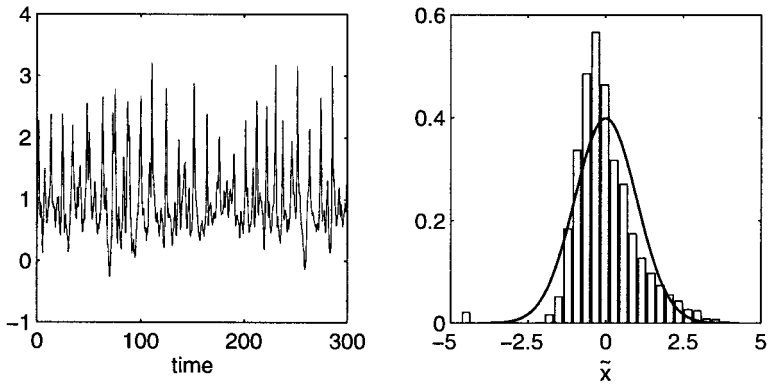

Fig. 2. Wave elevation.

Table 1
Topics of stochastic mechanics

	System (operator)	
Input	Deterministic	Stochastic
Deterministic	Solid mechanics	System reliability Homogenization
Stochastic	Random vibration	Stochastic finite element Seismic waves

have deterministic and stochastic coefficients for deterministic and stochastic systems, respectively. If the system and the input are deterministic, we deal with classical solid mechanics problems. We will see that some of these problems can be solved efficiently by methods of stochastic mechanics. If the system is deterministic and the input is random and time dependent, the output is a stochastic process. The theory of random vibration extending the methods of classical dynamics to the case in which the input is stochastic can be applied for solution. If the physical system is stochastic, the analysis is much more difficult because the algebraic/differential equations defining the system output have random coefficients.

2. A primer on probability

Our discussion will use some of the jargon of probability theory. This section presents a brief and heuristic review of the essential concepts of probability theory. Additional information on these concepts can be found, for example, in Grigoriu (1995), Lin and Cai (1995), Soong and Grigoriu (1993).

Histograms were already used in the previous section to justify the need for probabilistic models. Here, we define the concept of histogram by an example. Suppose that a record of the yearly maximum wind speed over a period of n years is available at a site, wind speed maxima in different years are unrelated, (a, b) denotes the range of the values of this record, and the wind speed range is divided in m equal sub-intervals. Let n_i be the number of data in the sub-interval i of (a, b). The *histogram* of the yearly maximum wind speed record gives the fraction of data n_i/n in each of the sub-intervals of (a, b). The ordinates of a histogram are positive and the sum of these ordinates over a sub-interval (a_1, b_1) in (a, b) provides an estimate of the probability that the yearly maximum wind speed takes values in (a_1, b_1). If the length n of the record approaches infinity and the partition of the interval (a, b) is refined, the histogram converges to a continuous function called *probability density function*. This function and the histogram have similar properties. Histograms can be developed for two or more data sets, for example, the records of yearly maximum wind speed and the corresponding wind direction. The resulting histogram is defined on $\mathbb{R}^2$ and converges to the *joint probability density function* of wind speed and direction as the sample size increases indefinitely. The joint probability density function is an extension of the probability density function to spaces of dimension two and higher.

The probability density function can be used to characterize *random variables*, that is, functions taking various values with specified probabilities. For example, consider the experiment of rolling a die characterized by the equally likely outcomes {1,2,3,4,5,6}. The outcomes of this experiment can be used to define a random variable X taking the values {1,2,3,4,5,6} with equal probability. The probability density function of X is $f(x) = (1/6)\delta(x-i)$, $i = 1,..., 6$, where δ denotes the delta function. This is not the only random variable that can be associated with the experiment of rolling a die. For example, consider a game in which we loose \$10 and win \$5 if the outcome is {1,2} (failure) and {3,4,5,6} (success), respectively. The random variable Y taking the values −\$10 and \$5 for the failure and success

events with probabilities 2/6 and 4/6, respectively, suffices to play this game. The random variable X can also be used to play the game but this variable provides unnecessary detailed information. If a random variable takes values in $\mathbb{R}^d$, $d \geq 2$, rather than on the real line, it is called a *random vector*. For example, consider the two-dimensional random vector $\mathbf{X} = (X_1, X_2)$ whose coordinates denote the number of hours a student studies for a course and the final grade. The random variables X_1 and X_2 are related so that the probability density functions of these variables are insufficient to characterize the random vector $\mathbf{X} = (X_1, X_2)$. The joint probability density function of (X_1, X_2) is needed to specify the law of $\mathbf{X}$.

Similar considerations can be used to introduce the concepts of stochastic process and field. For example, let $\{x_i(t)\}$, $i = 1,..., n$, be a set of n records consisting of hourly temperatures at, for example, the Cornell tower starting on January 1 and ending on December 31 of each year. The set of these records, called *sample paths*, can be used to characterize the trend and variability of temperature during a year at the Cornell tower. Consider a die with n equally likely sides corresponding to the available temperature records $\{x_i(t)\}$, $i = 1,..., n$, at the Cornell tower. To extract a record at random we need to roll this die and select the outcome, that is, one of the records $x_i(t)$. If n increases indefinitely, the resulting set of records defines a stochastic process $X(t)$ giving the temperature at the Cornell tower. A *stochastic process* can be viewed as a mathematical abstraction defining a rule for the random selection of sample paths from a set of specified functions of time. The characterization of stochastic processes is much more complicated than the characterization of random variables because we need to specify both the range of values with their likelihood as well as the time evolution the sample paths. The *stochastic field* resembles a stochastic process but depends on a space rather than a time argument. Because time has a natural flow and space does not, there are notable differences between these two concepts but these differences are not relevant for our discussion.

In many engineering problems a partial characterization of random quantities, referred to as the *second moment properties*, can provide useful information on the trend and the magnitude of the random fluctuations about this trend. The second moment properties consists of the *mean* and *variance* for random variables, the *mean vector* and *covariance matrix* for random vectors, and the *mean function* and *covariance functions* for stochastic processes and fields.

The mean and the arithmetic mean are similar concepts. For example, the mean μ of the random variable Y taking the values $-\$10$ and $\$5$ with probabilities 2/6 and 4/6, respectively, is $\mu = (-\$10)(2/6) + (\$5)(4/6) = 0$. The mean of a random vector $\mathbf{X}$ is equal to the vector defined by the means of its coordinates. The mean of the stochastic process giving the temperature at the Cornell tower can be approximated by the arithmetic mean, $\hat{\mu}(t) = (\frac{1}{n})\sum_{i=1}^{n} x_i(t)$, of the records $\{x_i(t)\}$. If $n \to \infty$, the estimate $\hat{\mu}(t)$ approaches the actual mean $\mu(t)$ of the process.

The variance of a random variable X is $\sigma^2 = \int (x-\mu)^2 f(x)\mathrm{d}x$ and resembles the concept of moment of inertia with f interpreted as material density. The variance of the random variable Y considered in the previous paragraph is $\sigma^2 = (-10-0)^2(2/6) + (5-0)^2(4/6) = 50$. The square root of the variance is called *standard deviation*. The ratio $v = \sigma/\mu$ is the *coefficient of variation* of Y and is defined for random variables with non-zero mean. The second moment properties of a random vector $\mathbf{X} = \{X_i\}$ consists of the mean μ_i, variance σ_i^2 of the random variables X_i, and a measure of the relationship between these random variables given by the *covariance*, $\gamma_{ij} = \iint (x_i-\mu_i)(x_j-\mu_j) f_{ij}(x_i, x_i)\mathrm{d}x_i\,\mathrm{d}x_j$ for all $i \neq j$, where f_{ij} denotes the joint probability density function of random variables X_i and X_j. The covariance γ_{ii} coincides with the variance σ_i^2 of X_i. If the covariance γ_{ij} are zero for all $i \neq j$, we say that the coordinates of the random vector $\mathbf{X}$ are *uncorrelated*. The mean vector $\boldsymbol{\mu} = \{\mu_i\}$ and the covariance matrix $\boldsymbol{\gamma} = \{\gamma_{ij}\}$ define the second moment properties of $\mathbf{X}$. The scaled covariance $\rho_{ij} = \gamma_{ij}/\sqrt{\gamma_{ii}\gamma_{jj}}$ is called the *correlation coefficient* of X_i and X_j. The covariance function of a stochastic process $X(t)$, denoted by $c(t, s)$, is equal to the covariance of the random variables $X(t)$ and $X(s)$ for all values of times t and s. If the mean function $\mu(t) = \mu$ is constant and the covariance function $c(t, s) = c(|t-s|)$ depends only on the time lag $|t-s|$, the process is said to be *weakly stationary*. The Fourier transform of $c(|t-s|)$ is called

the *mean power spectral density* of $X(t)$ and provides information on the frequency content of this process. The mean power spectral density can be viewed as the stochastic counterpart of the Fourier transform. A stochastic process with constant mean power spectral density is called *white noise*. The definitions of this paragraph can be extended without difficulties to stochastic processes and fields taking values in a vector space rather than the real line (Grigoriu, 1995; Soong and Grigoriu, 1993).

The class of *Gaussian* random variables, random vectors, and stochastic processes is particularly useful for both theoretical studies and applications. We say that an $\mathbb{R}^d$-valued random vector $\mathbf{X}$ with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\gamma}$ is Gaussian if its probability density function is

$$f(\mathbf{x}) = [(2\pi)^d \det(\boldsymbol{\gamma})]^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\gamma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \right], \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

where the superscript T denotes matrix transposition. A stochastic process $X(t)$ is said to be Gaussian if the random vectors consisting of values of the process at an arbitrary number of times is Gaussian. The Gaussian vectors and processes have the property that they remain Gaussian under linear transformations.

3. Deterministic operator and input

The solution of deterministic problems of equilibrium, propagation, and dynamics is the focus of classical solid mechanics. Generally, the solutions of these problems satisfy complex algebraic or partial differential equations that cannot be obtained in closed form. The finite difference, finite element, boundary element, and other numerical methods are used for solution. While these methods are extremely powerful and versatile, they do have certain drawbacks. A common feature of all these methods is that the solution must be calculated in the entire domain of definition of a problem, even if it is needed at a single point or a small number of points of the domain of definition.

It is possible to solve linear systems of equations, algebraic eigenvalue problems, homogeneous or inhomogeneous integral Fredholm equations, and other deterministic equations relevant to solid mechanics by methods of stochastic mechanics. For example, the Monte Carlo simulation was used to solve the Fredholm equation. This equation can be viewed as a recurrence formula relating current and updated approximations of the solution that involves an integral with integrand depending on a Green's function. The normalized Green function interpreted as a probability density function can be used to calculate this integral by Monte Carlo simulation and update the approximate solution. The method was applied to solve a large number of physics problems (Kalos and Whitlock, 1986). There have been no notable attempts to solve solid mechanics problems by Monte Carlo simulation or other methods of stochastic mechanics.

A current research trend in stochastic mechanics relates to the development of alternative methods for solving boundary value problems relevant to solid mechanics by Monte Carlo simulation, for example, the *random walk* and other methods (Grigoriu, 1997b; Sabelfeld, 1991). The random walk method is *local* in the sense that it gives the solution of a partial differential equation at an arbitrary point of the domain of definition directly rather than extracting it from the field solution. There is no need to formulate or solve linear systems of equations as required by the traditional numerical methods of solid mechanics. Although the theoretical considerations supporting the random walk and related methods are relatively complex and involve concepts of elasticity, applied mathematics, random processes, and stochastic integrals, the numerical algorithms for solution have attractive features. These algorithms are simple to program, always stable, accurate, local, and ideal for parallel computation.

This section illustrates the application and some features of the random walk method by a simple

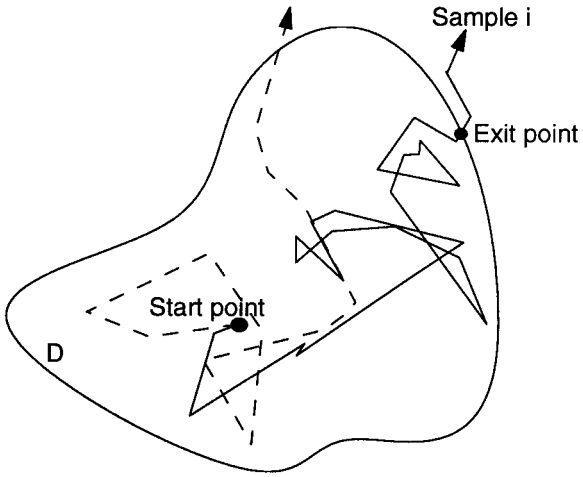

Fig. 3. Two samples of Brownian motion process $\mathbf{B}(t)$.

example. Let u be the solution of the Dirichlet boundary value problem

$$\Delta u(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in D,$$

$$u(\mathbf{x}) = h(\mathbf{x}),\ \mathbf{x} \in \partial D, \tag{2}$$

where D is an open bounded set in $\mathbb{R}^d$, ∂D denotes the boundary of D, $\Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$ defines the Laplace operator, and g; h are specified functions. The solution $u(\mathbf{x})$ can be interpreted as the steady state temperature at point $\mathbf{x}$ in a material of unit conductivity occupying the domain D subject to the flux g and boundary temperature h. The objective is to find the local solution of this boundary value problem, that is, the value of the unknown function u at an arbitrary point $\mathbf{x}$ of D. The classical methods of computational mechanics cannot deliver the value of $u(\mathbf{x})$ directly. As previously indicated, these methods have to extract the value of the solution u at $\mathbf{x}$ from the field solution.

The random walk method uses sample paths of an $\mathbb{R}^d$-valued Brownian motion process $\mathbf{B}(t)$ to estimate $u(\mathbf{x})$ at an arbitrary point $\mathbf{x}$ in D. Fig. 3 shows two sample paths of $\mathbf{B}(t)$ starting at an interior point $\mathbf{x}$ of a two-dimensional domain D. Let τ_k and $\mathbf{x}_k$ be the time at which sample k of $\mathbf{B}(t)$ leaves D for the first time and the mark on ∂D of this sample at time τ_k, respectively. It can be shown that $u(\mathbf{x})$ can be approximated by

$$\hat{u}(\mathbf{x}) = \frac{1}{n}\sum_{k=1}^{n} h(\mathbf{x}_k) - \frac{1}{2n}\sum_{k=1}^{n}\int_0^{\tau_k} g(\mathbf{b}_k(s))\mathrm{d}s, \tag{3}$$

where $\{\mathbf{b}_k(t)\}$, $k = 1,2,\ldots, n$, are samples of $\mathbf{B}(t)$ and n denotes the sample size. The proof of Eq. (3) involves technicalities that can be found in Chung and Williams (1990), Grigoriu (1997b), and Øksendal (1992). The approximation $\hat{u}(\mathbf{x})$ approaches the exact solution $u(\mathbf{x})$ as the sample size n approaches infinity. The approximate solution of Eq. (3) depends on averages of the boundary values of u and of the integral of the flux along the sample paths of the Brownian motion.

The formula of Eq. (3) can be used to implement a numerical algorithm for finding the local solution of Eq. (2). The implementation involves the generation of sample paths of the Brownian motion $\mathbf{B}(t)$, that we have not yet defined. The generation of samples of $\mathbf{B}(t)$ can be based on the fact that the

increments of this process over non-overlapping time intervals are independent Gaussian vectors with mean zero and covariance equal to the identity matrix $\mathbf{i}$ scaled by the square of the time increment (Grigoriu, 1995). For example, suppose that sample k of the Brownian motion process is at point ξ_k at time t. At a later time t' this sample is at $\xi_k + \mathbf{z}_k$, where $\mathbf{z}_k$ is a sample of a d-dimensional Gaussian vector with mean zero and covariance matrix $\mathbf{i}\sqrt{t'-t}$. Efficient algorithms are available for generating the Gaussian samples $\mathbf{z}_k$ (Grigoriu, 1995).

Consider for illustration the steady state solution of the heat equation for an eccentric annulus D of unit outer radius with a circular hole of radius 1/4 shifted from the center of the unit disc by 1/4. The boundary conditions are $u = 100$ and $u = 0$ for the outer and the inner boundaries of D, ∂D_1 and ∂D_2, and the flux g in Eq. (2) is assumed to be zero. The analytical solution of this problem can be found in Greenberg (1978) and is used to evaluate the accuracy of the random walk method. The approximate temperature at an arbitrary point $\mathbf{x}$ of D is $\hat{u}(\mathbf{x}) = (n_1/n)(100)$ by Eq. (3), where n_1 denotes the number of samples of the Brownian motion exiting D through ∂D_1. The average error recorded at $\mathbf{x} = (0.7, 0)$; (0.9, 0); (0, 0.25); (0, 0.5); and (0,0.75) is 2.79% for $n = 1000$ samples and a time steps in the range (0.0001, 0.001). The error can be reduced by increasing the sample size and/or reducing the time step (Grigoriu, 1997b).

The random walk method has been applied to solve a relatively broad range of solid mechanics problems. The list of solutions includes the Laplace, Poisson, transient heat, transport, and Schrödinger equations with Dirichlet and Neumann boundary conditions. The algorithm for solution is similar to the one used in our illustration but the Brownian motion may have to be replaced with diffusion processes depending on the structure of the differential operator of the solid mechanics problem (Grigoriu, 1998a, 1998b, 1997a, 1997b).

4. Deterministic operator and stochastic input

Stochastic mechanics problems specified by deterministic operators and random inputs have been studied extensively and continue to be investigated. The operators can be linear or nonlinear, differential or algebraic and the input may be described by stochastic processes and/or random variables. The main objective of most studies in this area of stochastic mechanics is the determination of the law, second moment properties, and other statistics of the solution of these problems. This objective includes reliability studies corresponding to the first time the state violates a design condition, fatigue, and other various failure modes as well as the stability of the motion of dynamic systems subjected to random input, referred to as *stochastic stability*. The possible combinations of the types of operators and inputs define a broad range of stochastic mechanics problems. The nonlinear memoryless transformation of a scalar input process is one of the simplest operator-input combination and is frequently encountered in applications, for example, the wind pressure is equal approximately to the square of the wind speed process. Memoryless transformations of Gaussian processes define a class of non-Gaussian processes, called *translation processes*, that is useful for many applications (Grigoriu, 1995). The *random vibration* problems correspond to deterministic differential operators with random process inputs and constitute an extension of classical dynamic problems to the case in which the input is random.

Random vibration is one of the oldest topics of stochastic mechanics. Early random vibration studies focused on the determination of the second moment properties of the output of relatively simple linear systems. In some studies, it was assumed incorrectly that the output of linear systems is Gaussian irrespective of the distribution of the input. Moments and other probabilistic characteristics have also been obtained for the output of nonlinear systems subjected to Gaussian white noise based on numerical solutions of the Fokker–Planck–Kolmogorov equation or heuristic approximations, for example, closure methods for solving the infinite hierarchy of moment equations associated with nonlinear systems.

Methods have also been developed for calculating the reliability of dynamic systems subjected to random load processes. Most results are for stationary Gaussian outputs and design conditions related to the first excursion of a response or damage index above critical threshold fatigue life (Bolotin, 1984; Crandall and Mark, 1963). The stability of the solution of random vibration problems has also been investigated based on the properties of the first few moments of the state (Bolotin, 1984; Crandall and Mark, 1963).

Current research trends include both *analytical* and *numerical solutions* of random vibration problems. The analytical studies are focused on the development of general methods of analysis that can deliver sample properties of the output processes rather then global properties such as the mean and covariance functions (Naess and Krenk, 1996) and are based on less simple mathematical concepts than used in the past. For example, these studies:

1. use properties of diffusion, Lévy, martingales, semimartingales, and other processes,
2. distinguish between the Gaussian, Poisson, and Lévy white noises, and
3. operate with stochastic integrals defined in the Itô sense that do not obey the rules of classical calculus.

A large number of studies provide techniques for finding the law of the output of linear systems subjected to non-Gaussian inputs (Grigoriu, 1995). Recent developments related to nonlinear dynamic systems include:

(a) approximate solutions based on stochastic averaging, equivalent linearization, and consistent closure methods (Ariaratnam, 1994; Naess and Krenk, 1996; Roberts and Spanos, 1990),
(b) partial differential equations for the characteristic function of the output of nonlinear systems subjected to Gaussian, Poisson, or Lévy white noise (Naess and Krenk, 1996), and
(c) stochastic stability studies using properties of products of random matrices and Lyapunov exponents for nonlinear system subjected to both Gaussian and Poisson white noise (Ariaratnam, 1994; Grigoriu, 1996; Lin and Cai, 1995; Naess and Krenk, 1996; Wedig, 1988).

In contrast to earlier studies on stochastic stability based on the behavior of the output moments that may provide no information on the stability of the output samples, the emphasis of most recent studies is on the sample properties of the output (Naess and Krenk, 1996).

One other major current research trust relates to the development of efficient and accurate numerical algorithms for solving general stochastic mechanics problems (Schuëller, 1997). These developments can be divided in two groups: *Monte Carlo simulation* and solutions of *partial differential equations* for output statistics. The Monte Carlo simulation method can be applied to solve both linear and nonlinear random vibration problems regardless of their complexity. Importance sample and other techniques are studied for improving the efficiency of the direct Monte Carlo simulation method. The direct Monte Carlo method involves three steps. First, sample paths of the input stochastic process need to be generated. Methods for generating such samples can be found in (Grigoriu, 1995). Second, methods of classical dynamics can be applied to calculate the outputs corresponding to the generated input samples. Third, statistics of the output process can be estimated from its sample paths calculated in the previous step. The Monte Carlo method is general but can be inefficient because it requires a large number of deterministic analyses (Schuëller, 1997). New accurate and efficient numerical algorithms continue to be developed for solving partial differential equations giving statistics of the output. These algorithms include new numerical methods for finding the probability density of a system output by solving the Fokker–Planck–Kolmogorov equation in higher dimensions as well as other partial differential equations for output descriptors, for example, the characteristic function of the output process (Naess and Krenk, 1996; Soize, 1994; Spencer and Bergman, 1993). The classical dynamics problems are specified by deterministic models for both the system and the input. If the input is allowed to be a

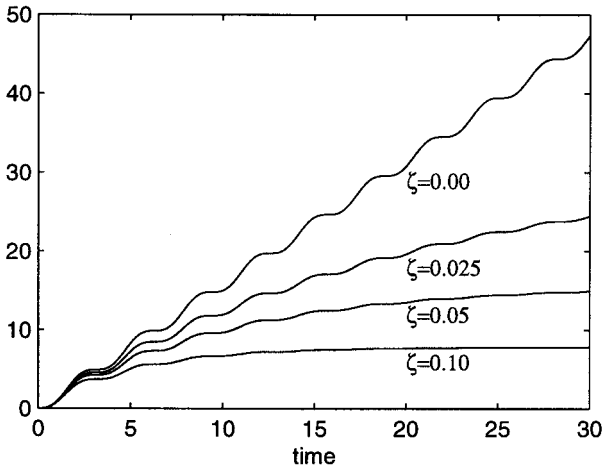

Fig. 4. Variance function of displacement $Y(t)$.

random process, the output is also a random process. The objective of the random vibration theory is to find statistics of the output of deterministic dynamic systems subjected to random excitation.

The degree of difficulty of the solution of random vibration problems depends on the system and input properties and the required output statistics. A simple example is used to demonstrate the solution of random vibration problems. The *linear random vibration* examines linear dynamic systems. Simple ordinary differential equations are available for calculating the second moment properties of the output these systems (Soong and Grigoriu, 1993). The differential equation for the output mean function coincides with the equation of motion of the system subjected to the mean function of input. However, the differential equation for the covariance function of output has no counterpart in classical dynamics. For example, consider a simple linear oscillator subjected to a weakly stationary white noise process $X(t)$ with mean zero and covariance function $c_x(t, s) = \pi g_0 \delta(|t-s|)$, where t and s are arbitrary times and the subscript x is used to indicate that this covariance function corresponds to $X(t)$. The mean power spectral density of this process is constant over all positive frequencies and equal to g_0. The oscillator displacement $Y(t)$ satisfies the differential equation of motion

$$\ddot{Y}(t) + 2\zeta\omega\,\dot{Y}(t) + \omega^2 Y(t) = X(t),\ t \geq 0, \tag{4}$$

where $\zeta \in (0,1)$ and $\omega > 0$ denote the damping ratio and the frequency of the system. Suppose that the oscillator is at rest at the initial time, that is, $Y(0) = 0$ and $\dot{Y}(0) = 0$. The mean of $Y(t)$ is zero at each time because $X(t)$ has mean zero and the initial conditions are equal to zero. The covariance and variance functions, $c_y(t, s)$ and $\sigma_y(t)^2 = c_y(t, t)$, of the output $Y(t)$ defined by Eq. (4) can be obtained in closed form (Soong and Grigoriu, 1993). For example, the displacement variance function is

$$\sigma_y(t)^2 = \frac{\pi g_0}{4\zeta\omega^3}\left[1 - \frac{1}{\beta^2}e^{-2\zeta\omega t}(\beta^2 + 2\zeta^2\omega^2 \sin^2(\beta t) + \zeta\omega\beta \sin(2\beta t))\right], \tag{5}$$

where $\beta = \omega\sqrt{1-\zeta^2}$. Fig. 4 shows the evolution of the variance function $\sigma_y(t)^2$ in the time range [0, 30]. As the time increases, $\sigma_y(t)^2$ approaches a constant value and the rate of convergence to this value increases with ζ. It can be shown that the asymptotic expression of the covariance function $c_y(t, s)$ for large times depends only on the time lag $|t-s|$ so that the oscillator displacement becomes a weakly stationary process as $t, s \to \infty$. If the input is a Gaussian process, the oscillator displacement is also

Gaussian and its second moment properties define the process $Y(t)$ completely. In this case, simple approximations are available for the probability that the maximum displacement during a specified time interval does not exceed a critical value (Grigoriu, 1995; Soong and Grigoriu, 1993). This probability provides a measure of the system performance useful for reliability analysis.

The solution of the linear random vibration problem complicates significantly if the input is non-Gaussian and second moment properties are insufficient for characterizing the output. Specialized techniques are needed to obtained output statistics other than the mean and covariance functions when the input is a non-Gaussian process (Grigoriu, 1995). Some of these techniques are based on the theory of nonlinear random vibration. The *nonlinear random vibration* examines nonlinear dynamic systems subjected to Gaussian and/or non-Gaussian excitation. As previously stated, there is no simple and general solution for nonlinear random vibration problems. A relatively large number of methods have been developed for analyzing nonlinear dynamic systems subjected to random excitation. The Fokker–Planck–Kolmogorov equations, moment equations, perturbation, stochastic averaging, equivalent linearization, and other methods can be used for solution (Grigoriu, 1995; Lin and Cai, 1995; Naess and Krenk, 1996; Roberts and Spanos, 1990; Schuëller, 1997). As mentioned previously, a major difficulty with the method of moments is that the moment equations form an infinite hierarchy that cannot be solved exactly (Lin and Cai, 1995; Soong and Grigoriu, 1993). For example, suppose that the linear restoring force $\omega^2 Y(t)$ of the oscillator defined by Eq. (4) is replaced by $\omega^2 Y(t)(1+\varepsilon Y(t)^2)$ and the mean of the input $X(t)$ is not zero, where ε is a small parameter. Then the mean displacement cannot be determined because the average of the equation of motion includes, in addition to the mean displacement and its derivatives, the mean of $Y(t)^3$ or the third order moment of $Y(t)$ that is not known. Similar situations are encountered with the differential equations for the higher order moments of the displacement process. Hence, the moment equations of $Y(t)$ form an infinite hierarchy that cannot be solved exactly. Heuristic assumptions relating higher and lower order moments of $Y(t)$ have been proposed to 'close' the moment equations (Soong and Grigoriu, 1993). Alternative solutions for closing the infinite hierarchy of moment equations that are not based on heuristic assumptions are also under development (Naess and Krenk, 1996).

5. Stochastic operator and deterministic input

Most physical *systems* consist of a large number of parts, referred to as *components*. The overall properties of a system depend on the properties of and the interaction among its components. Most component properties exhibit notable variations about their nominal values, for example, the measured strength, frequency of vibration, and fatigue life of a set of nominally identical beams differ from specimen to specimen. Probabilistic models are needed to capture the uncertainty in these properties. The *series* and *parallel systems* provide two extreme examples of component interaction. A series systems has no redundancy and fails by its weakest component or link. On the other hand, parallel systems have significant redundancy and their strength or capacity may depends in a complex way on the component mechanical properties and the way in which the surviving components share the applied load. The loading and environmental conditions can also influence or even determine some system properties. For example, the failure modes of a system under monotonic and cyclic loads may differ significantly.

The dimensions of the components of a system cover a broad range of scales from large parts of a system, for example, the wings of an aircraft and the beams and columns of a frame, to microscopic scales, for example, the phases of a multiphase medium and the grains of a polycrystal. A major objective of stochastic mechanics is the determination of *global properties* of a system based on the properties of and the interaction among its components. The focus of many early studies was the

approximate determination of the capacity of some physical systems and the development of phenomenological models for damage accumulation in materials. For example, a frequently used model for the capacity of brittle materials was based on the assumptions that

1. an arbitrary volume of a material contains a random number of weak spots,
2. failure occurs when the strength of the weakest spot is exceeded, that is, the failure condition for series systems, and
3. the capacities of the weak spots are independent identically distributed random variables (Bolotin, 1968; Freudenthal, 1961).

The resulting distribution for the system strength is Weibull and this distribution explains the observed decrease of the average strength of brittle materials with the specimen volume (Bolotin, 1968). Series systems can also be used to model statically determinate structures. Extensive studies were performed to determine the capacity of parallel systems with components of independent or dependent capacities $\{R_i\}$, $i = 1,..., n$. If the components are ideal elasto–plastic, the overall system strength is $R = \sum_{i=1}^{n} R_i$. If the system components are brittle, the determination of the overall system capacity is less simple because it requires to examine a large collection of equilibrium configurations with $n-m$, $m = 0, 1,..., n-1$, surviving components (Grigoriu, 1990; Phoenix, 1978). Parallel systems with brittle components are also called *Daniels* systems.

The development of phenomenological *damage accumulation models* for materials subjected to cyclic stresses is another research area investigated extensively in the past. Most studies in this area are based on mathematical models calibrated to experiments rather then physically-based models, for example, models based on the properties of the material microstructure. The *stress-independent* and *stress-dependent* models have been proposed to trace the evolution of a damage measure D in time for material subjected to cyclic loads (Bolotin, 1968). Consider a specimen subjected to a sequence of $\{n_k\}$ stress cycles of amplitudes $\{s_k\}$, $k = 1,..., m$, and define the damage measure $D = \sum_k (n_k/n_k^*)$, where n_k^* denotes the number of cycles at which the specimen fails under stress cycles of a constant amplitude s_k and is given by S–N curves. The specimen is said to survive if the damage measure does not exceed one and fails otherwise. Because n_k^* is uncertain, D is a random variable so that the surviving condition $D < 1$ can be satisfied with a probability, called *reliability*. This stress-independent damage accumulation model, known as *Miner's rule*, is inconsistent with the observations showing that the fatigue life depends on the order of application of the stress cycles. The stress-dependent models can account for the loading sequence but are difficult to calibrate to experimental results. The length A of a crack in a material subjected to cyclic loads is frequently used as a damage measure. The evolution of A in time can be given by a differential equation, for example, the *Paris model*, depending on some material constants, crack geometry, and stress intensity factor process (Madsen et al., 1986).

Stochastic systems with very small components constitutes another area of current research. *Homogenization* techniques are used to define equivalent homogeneous models for random heterogeneous media and develop consistent stochastic finite element methods. For example, these techniques give *effective* elastic constants for a continuum linear elastic homogeneous model that is equivalent with the actual material contained in a so-called *representative volume* consisting of a relatively large number of small components such as grains for aluminum or other metals. The equivalence is established from the condition that the homogeneous model behaves on average as the system of grains contained in the representative volume. The resulting elastic constants take values in a range defined by the *Voigt* and *Reuss* bounds depending on the boundary loading conditions. The dependence of the effective constants on the boundary loads decreases with the size of the representative volume (Ostoja-Starzewski, 1993; Willis, 1981). Homogenization is currently used to develop stochastic finite element formulations that are consistent with the material micromechanics (Ostoja-Starzewski, 1993). In these formulations, the size of the finite elements defines the size of the representative volume.

The study of *localization phenomena* at a broad range of scales is another research topic of current interest. For example, a detailed representation of the material microstructure is needed to capture localization phenomena at the microscopic scale, for example, nucleation and microcrack growth. Probabilistic models of grain geometry and atomic lattice orientation as considered in Arwade et al. (1998) can be used to model the material structure. Localization can also occurs in systems with macroscopic random components. For example, disordered cyclic systems, that is, nearly periodic dynamic systems with small random imperfections, exhibit modal localization (Lin and Cai, 1995; Xie and Ariaratnam, 1996). Several methods can be used to establish whether modal localization occurs and determine the shapes of the localized modes.

Two examples are presented to demonstrate the homogenization method and illustrate the uncertainty in the modal frequencies of linear systems with uncertain stiffness. First, we consider the determination of effective properties by homogenization. The effective properties of some systems can be obtained by elementary calculations. For example, the effective stiffness K_{eff} of a series and a parallel system with n linear elastic components of random stiffness $\{K_i\}$, $i = 1,..., n$, are $1/\sum_{i=1}^{n}(1/K_i)$. and $\sum_{i=1}^{n} K_i$, respectively. These formulas can be used to calculate probabilistic characteristics of K_{eff}. Suppose that $\{K_i\}$ are uncorrelated random variables with mean μ and coefficient of variation v. The mean and coefficient of variation of the stiffness of a parallel system with such components are $n\mu$ and $v/\sqrt{n}$, respectively, showing that the uncertainty in the system stiffness decreases with the number of components. Generally, the determination of effective material properties is rather complex. For example, consider a medium with random heterogeneous conductivity given by the random field $\Gamma(\mathbf{x})$, where $\mathbf{x}$ is an arbitrary point in the domain occupied by the material. The objective is to calculate the value of the effective conductivity γ_{eff} for an equivalent homogeneous medium model. The value of γ_{eff} can be obtained from the condition that the temperatures at the center of two spheres of radius $r > 0$ extracted from the random heterogeneous and the equivalent homogeneous media subjected to a constant flux coincide in the limit as $r \to \infty$ (Kim and Torquato, 1990). The determination of these temperatures requires to solve two heat equations corresponding to heterogeneous and homogeneous media. There are no analytical solutions of the heat equation for heterogeneous media so that the temperature at center of the random heterogeneous sphere has to be calculated numerically for a collection of samples of the conductivity random field $\Gamma(\mathbf{x})$. The resulting sample values of the temperature at the center of the heterogeneous sphere can be used to estimate γ_{eff}. The heterogeneous heat equation for each sample of $\Gamma(\mathbf{x})$ can be obtained by the finite element method. The approach can be inefficient because it involves the calculation of the field solution although the temperature is needed only at the center of the sphere and requires different finite element meshes for the samples of $\Gamma(\mathbf{x})$ to accommodate discontinuities in this field for multiphase media. The calculation efficiency can be improved for multiphase media with constant conductivity in each phase by using the Monte Carlo simulation method proposed in Kim and Torquato (1990). The method is based on estimates of the first exit time from a heterogeneous sphere of radius r for a Brownian motion starting at the center of the sphere and traveling with different speeds in various phases depending on the local conductivity. A difficulty with this method relates to the complex reflections exhibited by the Brownian motion at the phase interface. This difficulty limits the application of the method to multiphase materials with phases of relatively simple geometry. An alternative Monte Carlo solution related to the random walk method described in a previous section (Eqs. (2) and (3)) has been proposed in Grigoriu (1997a). The method can be applied to any type of heterogeneous material.

The second example examines the uncertainty in the eigenvalues of a matrix with random elements. Let $\mathbf{A}$ be a square symmetric matrix with random elements and consider the eigenvalue problem

$$\mathbf{AU} = \Lambda\mathbf{U}, \tag{6}$$

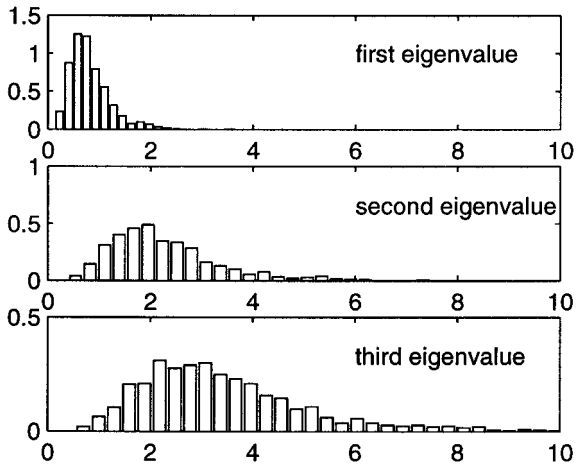

Fig. 5. Histograms of the eigenvalues of a three mass system.

where and Λ and $\mathbf{U}$ denote the random eigenvalues and eigenvectors of this matrix. Two classes of methods can be used for solution. If it is possible to obtain exact or approximate expressions for the eigenvalues Λ and the eigenvectors $\mathbf{U}$ of $\mathbf{A}$ as functions of the elements of this matrix, standard techniques can be used to find statistics of Λ and $\mathbf{U}$. Otherwise, specialized techniques need to be employed to obtained statistics of the eigenvalues and eigenvectors of $\mathbf{A}$, for example, the iteration, hierarchy, perturbation, crossing theory for random processes, Monte Carlo, and other methods (Boyce, 1968). The Monte Carlo method involves three steps. First, n samples of $\mathbf{A}$ need to be generated. Each sample of this matrix defines a deterministic eigenvalue problem. Second, deterministic algorithms can be used to find the eigenvalues and eigenvectors of each sample of $\mathbf{A}$. Third, histograms need to be developed for the eigenvalues and eigenvectors of $\mathbf{A}$ from the samples of these random quantities obtained in the previous step. Fig. 5 shows histograms of the eigenvalues of a discrete system with three equal masses connected in series by springs of random stiffness $\{K_i\}$, $i = 1, 2, 3$. The random stiffness vector $\mathbf{K} = (K_1, K_2, K_3)$ is defined by $\mathbf{K} = \exp(\mathbf{X})$, where $\mathbf{X}$ is a three dimensional Gaussian vector with mean $\boldsymbol{\mu} = (1, 1, 1)$, covariance matrix $\boldsymbol{\gamma} = \{\rho^{|i-j|}\}$, $i, j = 1, 2, 3$ and $\rho = 0.9$. The histograms correspond to $n = 1000$ samples of $\mathbf{K}$. Samples of the stiffness matrix $\mathbf{K}$ can be obtained by mapping the samples of the Gaussian vector $\mathbf{X}$ into samples of $\mathbf{K}$ according to the definition of this vector. Algorithms for generating Gaussian variables can be used to generate samples of $\mathbf{X}$ (Grigoriu, 1995). The figure shows that the range of the likely values of the eigenvalues of $\mathbf{A}$ increases with the order of these eigenvalues. The result can be used to explain difficulties related to the accurate measurement of the higher order frequencies of structural and mechanical systems.

6. Stochastic operator and input

Numerous mechanical and physical systems are characterized by uncertain properties, as described in the previous sections, and may be subjected to random rather than deterministic inputs. The output of these systems satisfies stochastic operators with random inputs and depends on two sources of uncertainty, the system and the input random properties. The stochastic operator can be linear or nonlinear, differential or algebraic. The input consists of random variables and/or stochastic processes.

The determination of (1) the law, second moment properties, or other statistics of the output or functions of the output and (2) the system reliability, that is, the probability that one or more output measures satisfy some design conditions during the system projected life τ, are the main objectives of most studies in this area. For example, it may be required to calculate the first two moments of the displacement of a dynamic system with uncertain parameters subjected to white noise or find the probability (reliability) that the crack trajectories originating at a location $\mathbf{x}_0$ of an aircraft do not leave a disk of specified radius centered at $\mathbf{x}_0$ during τ.

There is no general method for solving exactly a problem defined by a general stochastic operator and input. Exact solutions can be obtained only in a limited number of relatively simple cases. For example, suppose that the evolution of the variance of the displacement process Y given by Eq. (4) needs to be determined and the damping ratio ζ of this oscillator is uncertain. Let $f(z)$, $z \in (0, 1)$, be the probability density function of ζ. Accordingly, the differential operator of Eq. (4) is stochastic. The required variance function is given by the integral $\int \sigma_y(t; z)^2 f(z) \mathrm{d}z$ representing the expectation of the function $\sigma_y(t; z)^2$ given by Eq. (5) with $\zeta = z$. The solution of this problem is simple because the function $\sigma_y(t; z)^2$ is known and depends on a single random parameter. If the function $\sigma_y(t; z)^2$ depended on a large number of random parameters, that is, z was a random vector with a large number of coordinates, the calculation of the expectation of this function would be prohibitive by classical integration algorithms. Alternative integration algorithms based on, for example, the Monte Carlo simulation and/or the response surface methods may have to be used for solution (Madsen et al., 1986; Naess and Krenk, 1996).

Most of the early work on the analysis of mechanics problems with stochastic operators and inputs focused on the development of approximate analytical methods for solution and algorithms for reliability analysis. The approximate analytical methods include *perturbation, equivalent linearization, Neumann series expansions, decomposition series*, and other techniques (Adomian, 1986; Bolotin, 1968). Most of these methods separate the operator in a deterministic and a stochastic component and view the stochastic component of the operator as an additional random input. If the stochastic component has a small order ε, the perturbation method applies so that the solution can be expanded in a power series of ε and the terms of this series can be calculated sequentially as solutions of differential equations with the same operator, the deterministic component of the stochastic operator, but different random inputs (Adomian, 1986; Bolotin, 1968). For example, the perturbation method is used in Bolotin (1968) to calculate the second moment properties of the displacement of a beam with random imperfections supported by a Winkler elastic foundation. The past developments of *reliability methods* are particularly useful for evaluating the performance of large systems with uncertain properties subjected to random inputs. Numerical algorithms based on these developments, such as FORM/SORM (Madsen et al., 1986), are incorporated in computer codes and used for the probabilistic design of mechanical, aerospace, and offshore structures.

Current research directions include the development of *analytical methods* for solving some classes of stochastic mechanics problems and *efficient numerical algorithms* for analyzing general mechanics problems defined by stochastic operators and inputs. The analytical studies cover a broad range of problems, such as damage evolution in microstructure, formation of patterns in granular materials, and system reliability. For example, recent studies show that the density of immobile dislocations in grains subjected to plastic strain satisfies a Fokker–Planck equation and the solution of this equation can be used to estimate the pattern of dislocations (Braasch et al., 1996). Localization phenomena identified by the formation of patterns can be observed in granular material with position dependent stiffness. The occurrence of these phenomena is captured by some of the recent analytical developments (Koenders, 1998). Relatively simple solutions were also found recently for the reliability of Daniels systems with a large number of random components subjected to dynamic Gaussian inputs (Rychlik and Grigoriu, 1992).

Recent developments of efficient numerical algorithms for solving general stochastic mechanics problems are based on Monte Carlo simulation, stochastic finite and boundary element, finite differences, stochastic Green's functions, and other methods. The Monte Carlo simulation method is the most general approach and can be applied to solve any stochastic problem for which there is a numerical or analytical solution of the associated deterministic problem corresponding to arbitrary but fixed values of the random parameters. A drawback of the method is the potential inefficiency. The Monte Carlo method may require to solve a large set of deterministic problems corresponding to generated samples of the differential operator and input. The resulting output samples can be used to calculate the law or other response statistics and estimate reliability. For example, an estimate of the reliability of the crack growth problem discussed previously is given by the fraction of crack trajectories that do not leave a 'safe' domain during the system projected lifetime. If the system reliability is very high, a large number of crack trajectories needs to be generated to obtain a satisfactory estimate of reliability. Consider a loaded coin with the probabilities $1-p$ and $0 < p \ll 1$ of head (success) and tail (failure), respectively. The average number of tails in n tossing is np. Suppose that p is unknown and the objective is to estimate this probability from the outcomes of n trials. A value of np of at least ten is needed to estimate p satisfactorily so that the coin has to be tossed more than $10/p$ times, for example, $n = 10^5$ for $p = 10^{-4}$. In the contest of Monte Carlo simulation solution of the crack growth problem, at least 10^5 deterministic analyses have to be performed to estimate a reliability of order $1–10^{-4}$. In most practical cases, this number of calculations is prohibitive.

New applications of the Monte Carlo and other numerical solutions are currently under development. For example, stochastic Green's functions began to be used to generate *artificial seismic ground motions*. The input consists of probabilistic models for the seismic sources affecting the seismicity at a site. The system consists of the earth layers between seismic sources and site so that it can be viewed as a random heterogeneous medium. The propagation and scattering of waves from seismic sources to site can be described by stochastic Green's functions (Papageorgiou, 1997; Sato and Fehler, 1998; Sobczyk, 1985). These developments are essential for earthquake engineering particularly at sites with few or no seismic records, for example, the Eastern United States.

An elementary illustration of the *stochastic finite difference* method is presented here in some details. Additional information on the stochastic finite element related methods can be found, for example, in Ghanem and Spanos (1991). Consider a linear elastic system modeled by finite elements or differences with the vector of nodal displacements $\mathbf{U}$ defined by the algebraic equation

$$\mathbf{K}(\mathbf{X})\mathbf{U} = \mathbf{F}(\mathbf{X}), \tag{7}$$

where $\mathbf{K}$, $\mathbf{F}$, and $\mathbf{X}$ denote the stiffness matrix, the input vector, and a vector including all the random parameters of both the system and the input, respectively. The notations in this equation indicate that $\mathbf{K}$ and $\mathbf{F}$ depend on the vector of random parameters $\mathbf{X}$. Because $\mathbf{U}$ is the solution of Eq. (7), it also depends on $\mathbf{X}$ so that we can write $\mathbf{U} = \mathbf{U}(\mathbf{X})$. Suppose that the objective is to determine the second moment properties of $\mathbf{U}$, that is, the mean vector and the covariance matrix of $\mathbf{U}$. The hierarchy, perturbation, decomposition, response surface, first or higher order expansion, Monte Carlo simulation, and other methods can be applied for solution (Shinozuka and Deodatis, 1988). The method based on the first order Taylor expansion of $\mathbf{U}$ about the mean of $\mathbf{X}$ is used for illustration.

Let $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ be the mean and the covariance matrices of the vector of uncertain parameters $\mathbf{X}$. Consider the first order Taylor expansion

$$\mathbf{U}(\mathbf{X}) \simeq \mathbf{U}(\boldsymbol{\mu}) + \sum_p \frac{\partial \mathbf{U}(\boldsymbol{\mu})}{\partial x_p}(X_p - \mu_p) \tag{8}$$

of $\mathbf{U}$ about the mean $\boldsymbol{\mu}$ of $\mathbf{X}$. The approximate mean and covariance matrices of the output $\mathbf{U}$ are

$$\tilde{\boldsymbol{\mu}}_u = \mathbf{U}(\boldsymbol{\mu}),$$

$$\tilde{\boldsymbol{\gamma}}_u = \sum_{p,q} \frac{\partial \mathbf{U}(\boldsymbol{\mu})}{\partial x_p} \frac{\partial \mathbf{U}(\boldsymbol{\mu})}{\partial x_q} \gamma_{pq} \tag{9}$$

and depend on the value of the solution $\mathbf{U}$ and of its gradient for $\mathbf{X}$ equal to $\boldsymbol{\mu}$. The solution of the deterministic problem $\mathbf{K}(\boldsymbol{\mu})\mathbf{U} = \mathbf{F}(\boldsymbol{\mu})$ gives $\mathbf{U}(\boldsymbol{\mu})$ and requires to calculate the inverse $\mathbf{K}(\boldsymbol{\mu})^{-1}$ of the deterministic stiffness matrix $\mathbf{K}(\boldsymbol{\mu})$. The components of the gradient of $\mathbf{U}$ at $\boldsymbol{\mu}$ can be calculated from the equations

$$\frac{\partial \mathbf{U}(\boldsymbol{\mu})}{\partial x_p} = \mathbf{K}(\boldsymbol{\mu})^{-1} \left(\frac{\partial \mathbf{F}(\boldsymbol{\mu})}{\partial x_p} - \frac{\partial \mathbf{K}(\boldsymbol{\mu})}{\partial x_p} \mathbf{U}(\boldsymbol{\mu}) \right) \tag{10}$$

obtained from Eq. (7) by differentiation. The calculations of the gradients of $\mathbf{U}$ in Eq. (10) involves the inversion of the stiffness matrix $\mathbf{K}(\boldsymbol{\mu})$ that was previously obtained to find the mean solution.

Consider for illustration a beam on elastic foundation of span l fixed and simply supported at the left and right end, respectively. The beam has a constant stiffness EI and is subjected to a uniformly distributed load of random intensity X_1. The foundation stiffness is random and can be modeled by a stochastic field $Z(\xi)$, $\xi \in (0, l)$. The displacement $U(\xi)$ of the beam satisfies the differential equation $\mathrm{d}^4U(\xi)/\mathrm{d}\xi^4 + (Z(\xi)/EI)U(\xi) = X_1/(EI)$ with the boundary conditions $U(0) = 0$, $\mathrm{d}U(0)/\mathrm{d}\xi = 0$, $U(l) = 0$, and $\mathrm{d}^2U(l)/\mathrm{d}\xi^2 = 0$. The finite difference solution of this problem for nodes equally spaced at $l/4$ leads to the linear algebraic equation:

$$\begin{bmatrix} 7 + aX_2 & -4 & 1 \\ -4 & 6 + aX_3 & -4 \\ 1 & -4 & 5 + aX_4 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} aX_1 \\ aX_1 \\ aX_1 \end{bmatrix}, \tag{11}$$

defining the beam displacements U_1; U_2; U_3 at $\xi = l/4$; $l/2$; $3l/4$, where $a = (l/4)^4/(EI)$ and X_1, $X_2 = Z(l/4)$, $X_3 = Z(l/2)$, $X_4 = Z(3l/4)$ are the coordinates of the vector of random parameters $\mathbf{X}$. The gradients of the matrices $\mathbf{F}$ and $\mathbf{K}$ for this problem (7) can be obtained simply. For example, the vector $\partial \mathbf{F}(\boldsymbol{\mu})/\partial x_p$ is $a\mathbf{1}$ for $p = 1$ and $\mathbf{0}$ for $p \neq 1$, the elements of $\partial \mathbf{K}(\boldsymbol{\mu})/\partial x_1$ are zero and the elements of $\partial \mathbf{K}(\boldsymbol{\mu})/\partial x_2$ are zero except for the element (1, 1) that is equal to a. Simple calculations using Eqs. (8)–(10) give the approximate means $\tilde{\boldsymbol{\mu}}_u = (0.285, 0.457, 0.373)$ in., standard deviations (0.0688, 0.967, 0.1061) in. and correlation coefficients $\tilde{\rho}_{12} = 0.8946$; $\tilde{\rho}_{13} = 0.9127$; $\tilde{\rho}_{23} = 0.8874$ for $E = 1.5 \times 10^6$ lbs/in^2, $I = 430$ in^4, $\mu_1 = 1000$ lbs/in; $v_1 = 0.20$, $\mu_p = 2000$ lbs/in^2; $v_p = 0.30$ for $p = 2, 3, 4$, and uncorrelated coordinates of $\mathbf{X}$.

7. Conclusions

The properties of many physical systems and/or the actions on these systems exhibit notable random fluctuations that cannot be captured by deterministic models. Examples were used to demonstrate the need for probabilistic models to represent systems and inputs depending on uncertain parameters. The displacement, stress, strain, damage, and other output measures of these systems are stochastic so that their properties cannot be obtained by the deterministic methods of classical solid mechanics. The methods of stochastic mechanics are needed for solution. Following an introduction and a section on elementary concepts of probability theory, the four possible combinations of deterministic/stochastic

system and input were examined in separate sections. The case of deterministic systems and inputs is discussed to demonstrate the potential of the stochastic methods for solving classical problems of solid mechanics. The case of deterministic operators and random inputs is focused on random vibration problems because these problems constitute a major topic of stochastic mechanics. The last two sections deal with problems defined by stochastic operators and deterministic or stochastic inputs. All the sections on stochastic mechanics review some of the most notable past accomplishments, outline current research trends on both analytical and numerical solutions, and illustrate some of these research trends by simple examples.

References

Adomian, G., 1986. Nonlinear Stochastic Operator Equations. Academic Press, New York.

Ariaratnam, S.T., 1994. Some illustrative examples of stochastic bifurcation. In: Thomson, J.M.T., Bishop, S.R. (Eds.), Nonlinearity and Chaos in Engineering Dynamics. John Wiley & Sons, New York, pp. 267–274.

Arwade, S., Grigoriu, M., Ingraffea, A.R., Miller, M., 1998. Crack growth in stochastic microstructures. In: 4th International Conference on Stochastic Structural Dynamics.

Bolotin, V.V., 1968. Statistical Methods in Structural Mechanics. Holden–Day, Inc, San Francisco.

Bolotin, V.V., 1984. Random Vibration of Elastic Systems. Martinus Nijhoff Publishers, Boston.

Boyce, W.E., 1968. Random eigenvalue problems. In: Probabilistic Methods in Applied Mathematics, vol. 1. Academic Press, New York, pp. 2–73.

Braasch, H., Estrin, Y., Bréchet, Y., 1996. A stochastic model for dislocation density evolution. Scripta Materialia 35, 279–284.

Chung, K.L., Williams, R.J., 1990. Introduction to Stochastic Integration. Birkhäuser, Boston.

Crandall, S.H., Mark, W.D., 1963. Random Vibration in Mechanical Systems. Academic Press.

Freudenthal, A.M., 1961. Reliability analysis of fatigue-sensitive structures. Acta Technica, 55–81.

Ghanem, R.G., Spanos, P.D., 1991. Stochastic Finite Element: A Spectral Approach. Springer–Verlag, New York.

Greenberg, M.D., 1978. Foundations of Applied Mathematics. Prentice Hall, Inc, Englewood Cliffs, New Jersey.

Grigoriu, M., 1990. Reliability analysis of dynamic Daniels systems with local load sharing rule. Journal of Engineering Mechanics 116 (12), 2625–2642.

Grigoriu, M., 1995. Applied Non-Gaussian Processes. Examples, Theory, Simulation, Linear Random Vibration, and MATLAB Solutions. Prentice Hall, Englewood Cliffs, New Jersey.

Grigoriu, M., 1996. Lyapunov exponents for nonlinear systems with Poisson white noise. Physics Letters A 217, 258–262.

Grigoriu, M., 1997a. Estimation of effective conductivity of random heterogeneous media by diffusion processes. Journal of Applied Physics 82, 4346–4349.

Grigoriu, M., 1997b. Solution of some elasticity problems by the random walk method. ACTA Mechanica 125 (1–4), 197–209.

Grigoriu, M., 1998a. A local solution of the Schrödinger equation. Journal of Physics A: Mathematical and General 31, 8669–8676.

Grigoriu, M., 1998b. A Monte Carlo solution of transport equation. Probabilistic Engineering Mechanics 13 (3), 169–174.

Kalos, M.H., Whitlock, P.A., 1986. Monte Carlo Methods. John Wiley & Sons, New York.

Kim, I.C., Torquato, S., 1990. Determination of the effective conductivity of heterogeneous media by Brownian motion simulation. Journal of Applied Physics 68, 3892–3903.

Koenders, M.A., 1998. Effects of evolution on a heterogeneous material. In: Proceedings of the 12th Engineering Mechanics Conference, pp. 32–35.

Lin, Y.K., Cai, G.Q., 1995. Probabilistic Structural Dynamics. McGraw–Hill, Inc, New York.

Madsen, H.O., Krenk, S., Lind, N.C., 1986. Methods of Structural Safety. Prentice–Hall, Inc, Englewood Cliffs, NJ.

Naess, A., Krenk, S. (Eds.), 1996. Advances in Nonlinear Stochastic Mechanics. Kluwer Academic, Boston.

Øksendal, B., 1992. Stochastic Differential Equations. Springer–Verlag, New York.

Ostoja-Starzewski, M., 1993. Micromechanics as a basis of random elastic continuum approximations. Probabilistic Engineering Mechanics 8, 107–114.

Papageorgiou, A., 1997. Engineering seismology. In: Beskos, D.E., Kausel, E. (Eds.), Advances in Earthquake Engineering, vol. 3.

Phoenix, S.L., 1978. The stochastic strength and fatigue of fiber bundles. International Journal of Fracture 14, 327–344.

Roberts, J.B., Spanos, P.D., 1990. Random Vibration and Statistical Linearization. John Wiley & Sons, New York.

Rychlik, I., Grigoriu, M., 1992. Reliability of Daniels systems with equal load sharing rule subject to stationary Gaussian dynamic loads. Probability Engineering Mechanics 7, 113–121.

Sabelfeld, K.K., 1991. Monte Carlo Methods in Boundary Value Problems. Springer–Verlag, New York.

Sato, H., Fehler, M.C., 1998. Seismic wave propagation and scattering in the heterogeneous Earth. In: Modern Acoustics and Signal Processing.

Schuëller, G.I., 1997. A state-of-the-art report on computational stochastic mechanics. Probabilistic Engineering Mechanics 12 (4), 199–321.

Shinozuka, M., Deodatis, G., 1988. Response variability of stochastic finite element systems. Journal of Engineering Mechanics, ASCE 118 (3), 499–519.

Sobczyk, K., 1985. Stochastic Wave Propagation. Elsevier, New York.

Soize, C., 1994. The Fokker–Planck Equation for Stochastic Dynamical Systems and its Explicit Steady State Solution. World Scientific, New Jersey.

Soong, T.T., Grigoriu, M., 1993. Random Vibration of Mechanical and Structural Systems. Prentice Hall, Englewood Cliffs, N.J.

Spencer, B.F., Bergman, L.A., 1993. On the numerical solution of the Fokker–Planck equation for nonlinear stochastic systems. Journal of Nonlinear Dynamics 4, 357–372.

Wedig, W.V., 1988. Lyapunov Exponents of Stochastic Systems and Related Bifurcation Problems. Elsevier Applied Science, New York, pp. 315–327.

Willis, J.R., 1981. Variational and related methods for the overall properties of composites. Advances in Applied Mechanics 21, 1–78.

Xie, W.-C., Ariaratnam, S.T., 1996. Vibration mode localization in disordered cyclic structures, II: Multiple substructure modes. Journal of Sound and Vibration 189, 647–660.